Relativity and gravitation

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Abstract

This article specifies some aspects of calculations involved in general relativity, whose consideration is often approximate and can alas sometimes lead to misinterpretations.

Introduction

In its article "Relativity of space-time, gravitational effects" [1], C. Geuting lets remain some ambiguities of notations which it seems useful to mention.

1 Space-time in the presence of gravitation

1.1 Notation for the metric

We proceed in the presence of a star of mass M, motionless and of spherical symmetry. The space-time metric can be written with the "classic" notations in spherical coordinates:

$$ds^2 = A(r)\ c^2dt^2 - B(r)\ dr^2 - r^2\ d\theta^2 - r^2 sin^2(\theta)\ d\phi^2. \tag{1}$$

1.2 Motion of a free particle in the gravitational field

The length of a line of universe followed by the particle between P_1 and P_2 can be written, using a parameterization by a variable u which may a priori seem "unspecified" and while noting $X^\bullet = \dfrac{dX}{du}$:

$$\begin{aligned} s_{12} &= \int_{s_1}^{s_2} ds = \int_{u_1}^{u_2} \frac{ds}{du}\ du \\ &= \int_{u_1}^{u_2} \sqrt{A(r)\ c^2 t^{\bullet 2} - B(r)\ r^{\bullet 2} - r^2\ \theta^{\bullet 2} - r^2 sin^2(\theta)\ \phi^{\bullet 2}}\ \ du. \end{aligned}$$

According to Fermat's principle, this can be interpreted in the form:

$$s_{12} = \int_{s_1}^{s_2} L(u, t, r, \theta, \phi, t^\bullet, r^\bullet, \theta^\bullet, \phi^\bullet)\ du$$

with a Lagrangian not depending explicitly on u:

$$L(t, r, \theta, \phi, t^\bullet, r^\bullet, \theta^\bullet, \phi^\bullet) = \sqrt{A(r)\ c^2 t^{\bullet 2} - B(r)\ r^{\bullet 2} - r^2\ \theta^{\bullet 2} - r^2 sin^2(\theta)\ \phi^{\bullet 2}}.$$

The solutions $t(u)$, $r(u)$, $\theta(u)$ and $\phi(u)$ satisfy the equations of Euler-Lagrange:

$$\frac{d}{du}\frac{\partial L}{\partial t^\bullet} = \frac{\partial L}{\partial t} \quad ; \quad \frac{d}{du}\frac{\partial L}{\partial r^\bullet} = \frac{\partial L}{\partial r} \quad ; \quad \frac{d}{du}\frac{\partial L}{\partial \theta^\bullet} = \frac{\partial L}{\partial \theta} \quad ; \quad \frac{d}{du}\frac{\partial L}{\partial \phi^\bullet} = \frac{\partial L}{\partial \phi}.$$

But in these conditions, is the variable u really "unspecified" as indicated in [1]? In fact not since when postponing the preceding solutions in the relation (1) we deduce $s(u)$. In fact we obtain $L = \frac{ds}{du} = Cst$, which corresponds to an expression $s(u)$ affine (for a massive particle, one could have parameterized by s).

It is moreover this property, once it is known, which allows faster calculations by replacing L by its square (the effect on the result is the same as multiplying by a constant). Otherwise, it would be impossible to justify the property $s_{12} = \int_{s_1}^{s_2} ds = \int_{u_1}^{u_2} \frac{ds}{du}\ du \ \propto \int_{u_1}^{u_2} \Big(\frac{ds}{du}\Big)^2 du.$

The particular case of the photons corresponds to the case $s(u) = Cst$, in accordance with the characteristic property $ds = 0$. Moreover conversely it is this case, impossible to parameterize by s, which is the most simplified by this method. The analogy with non-relativistic mechanics and special relativity shows in fact that the quantity to be minimized for a massive particle is the action $S_{12} = \int_{s_1}^{s_2} mc\ ds$. When comparing the momentums, the passage to the limit for zero mass shows that it is necessary to replace $\frac{mc}{s^\bullet}$ by $\frac{h\nu}{c}$, which requires to introduce an additional $\frac{ds}{du}$ factor: when replacing L by its square one automatically removes the difficulty of the passage to the limit and one can treat in the same way massive or massless particles [2].

One thus obtains [1], for $\theta = \frac{\pi}{2}$ (that it is always possible to impose by a choice of axes orientation):

$$\frac{r^2}{A(r)}\frac{d\phi}{dt} = J = Cst; \tag{2}$$

$$\frac{B(r)}{A^2(r)}\Big(\frac{dr}{dt}\Big)^2 + \frac{r^2}{A^2(r)}\Big(\frac{d\phi}{dt}\Big)^2 - \frac{c^2}{A(r)} = H = Cst; \tag{3}$$

with in addition, while posing $R_s = \dfrac{2GM}{c^2}$:

$$A(r) = 1 - \frac{R_s}{r} = \frac{1}{B(r)}. \tag{4}$$

2 Application to black holes

2.1 Singularity of the metric

The expression (4) highlights the singular behavior of the metric: for $r < R_s$ one obtains $A(r) < 0$ and the coordinate t seems to be no more a temporal variable. This is associated with the properties of the stars named "black holes".

It can be then useful to reconsider the problem with other coordinates, in particular with the "isotropic" metric:

$$ds^2 = \underline{A}(\underline{r})\ c^2 dt^2 - \underline{B}(\underline{r})\ \Big(d\underline{r}^2 + \underline{r}^2\ d\theta^2 + \underline{r}^2 sin^2(\theta)\ d\phi^2\Big). \tag{5}$$

One can determine $\underline{A}(\underline{r})$ and $\underline{B}(\underline{r})$ by solving of the equations of the field [3, 4], but by comparison with (1), this change of radial coordinate imposes:

$$\underline{B}(\underline{r})d\underline{r}^2 = B(r)dr^2 \quad and \quad \underline{B}(\underline{r})\underline{r}^2 = r^2\ ;$$

from that one can deduce:

$$\frac{d\underline{r}^2}{\underline{r}^2} = B(r)\frac{dr^2}{r^2}\ ; \qquad \frac{d\underline{r}}{\underline{r}} = \pm\frac{dr}{r.(r - R_s)}\ ;$$

$$\underline{r} = r\,.\frac{\Big(1 - \frac{R_s}{2r}\Big) \pm \sqrt{1 - \frac{R_s}{r}}}{2}\ ; \tag{6}$$

or conversely, while noting $\underline{R}_s = \dfrac{R_s}{4}$:

$$r = \underline{r}\,.\Big(1 + \frac{\underline{R}_s}{\underline{r}}\Big)^2. \tag{7}$$

This then makes it possible to write:

$$\underline{B}(\underline{r}) = \Big(1 + \frac{\underline{R}_s}{\underline{r}}\Big)^4\ ; \quad \underline{A}(\underline{r}) = \Big(\frac{\underline{r} - \underline{R}_s}{\underline{r} + \underline{R}_s}\Big)^2. \tag{8}$$

It is then useful to notice that the singularity for $r = R_s$ corresponds to that for $\underline{r} = \underline{R}_s$, but that $\underline{A}(\underline{r})$ cancels without becoming negative: the coordinate t remains a temporal variable.

This difference in behavior seems inconsistent with the physical properties. The quantity A is a property associated with the point P in space (fixed) where it is calculated; expressions $A(r)$ and $A(\underline{r})$ differ because the marking of P is not made in the same way, but the result is the same A (this is why we do not note $\underline{A}$). Unambiguously for a static metric, the flow of time at P does not depend on the way in which one locates the space position of P.

The reason of this apparent difference comes from the non-bijective character of the relations (6) and (7): for $\underline{r} < \underline{R}_s$ (inside the singular zone) one obtains $r > R_s$ (*cf.* figure 1). In fact, the "radius" r is determined from the perimeter, but because of the strong curvature of space some spheres can have an "inner radius" (distance to the center) smaller and yet a "peripheral radius" larger. Thus inside one obtains $A(r) > 0$ and the coordinate t remains indeed a temporal variable (as well as with isotropic coordinates) [5].

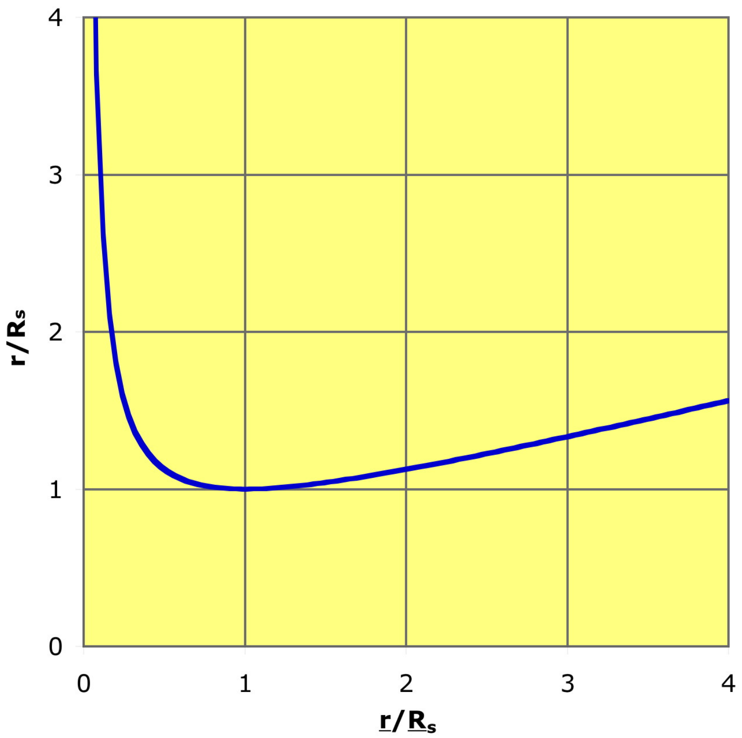

Fig. 1: variations (in reduced notations) of the "classic" radial variable as a function of the "isotropic" radial variable.

2.2 Freefall of a massive particle

Let us consider a massive particle, released with a null initial speed outside the singular zone (for the reasons evoked in the preceding part, we carefully avoid here noting that by $r > R_s$).

In this case, the equation (3) can be written:

$$\frac{B(r)}{A(r)}\Big(\frac{dr}{dt}\Big)^2 - c^2 = H\ A(r). \tag{9}$$

This gives under the initial conditions: $-c^2 = H\ A(r_0)$; then the norm of the falling speed (established with the classic r coordinate, but valid even with the $\underline{r}$ coordinate):

$$v = \sqrt{\frac{B}{A}}\ \left|\frac{dr}{dt}\right| = c\ \sqrt{1 - \frac{A}{A_0}}\ . \tag{10}$$

An unexpected consequence is that, if the speed increases until the singularity (it tends towards c when A approaches 0), it afterwards decreases once in the inner zone (A increases): the gravitational force is repulsive [6]. Or, in other words, the gravitation trains in the direction where the field lines approach one another, but in this area the curvature of space is such that it corresponds to a drive towards outside.

As stated in [1], the duration of fall until the singularity seems infinite for any motionless observer located outside. To verify that the duration of fall is finished for the particle in freefall, one can write: $B\ dr^2 = \Big(1 - \frac{A}{A_0}\Big) A\ c^2 dt^2$, and then deduce: $ds^2 = c^2 d\tau^2 = A\ c^2 dt^2 - B dr^2 = \frac{A^2}{A_0}\ c^2 dt^2$. One thus obtains (valid in the same way with the coordinate $\underline{r}$):

$$\Big(\frac{dr}{d\tau}\Big)^2 = c^2.\Big(\frac{A_0 - A}{AB}\Big). \tag{11}$$

With the classic r coordinate, the equation simplifies and seems without problem for the crossing of the singularity:

$$\Big(\frac{dr}{d\tau}\Big)^2 = c^2.\Big(\frac{R_s}{r} - \frac{R_s}{r_0}\Big). \tag{12}$$

One can while derivating also obtain the form:

$$\frac{d^2 r}{d\tau^2} = -\frac{c^2 R_s}{2r^2}. \tag{13}$$

This form is more practical to integrate, because while carrying out into (12) the initial conditions: $r = r_0$ et $\frac{dr}{d\tau} = 0$, one obtains the constant mathematical solution, clearly unphysical (this difficulty is avoidable, but complicates unnecessarily).

Part of the simplification comes from the fact that for the crossing $v = c$ and $d\tau = 0$, but that in addition $dr = 0$ since this variable r passes by a minimum. A hidden difficulty is that $\frac{dr}{d\tau}$ changes its sign when crossing the singularity.

Numerical integration without precaution gives a curve shown in figure 2 and seeming to pass the singularity with values $r < R_s$ (part in dotted lines). But the correct curve must be traced with $r > R_s$ (part in dashes; it turns out that this corresponds to a discontinuity, but that's only because the variable r is not necessarily judicious for this case).

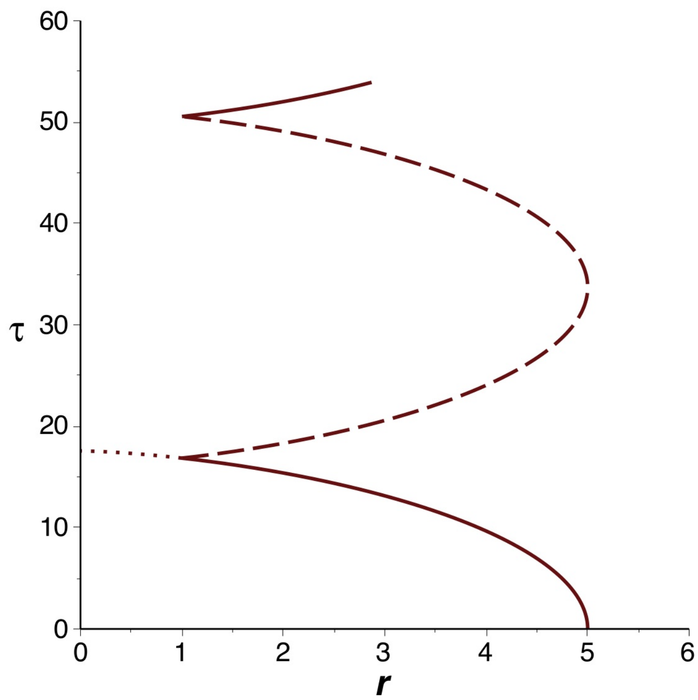

Fig. 2: variations (in reduced notations) of the proper duration of fall as a function of the "classic" radial variable.

We check that inside the particle reaches the limiting coordinate $r = r_0$ (if it does not reach before the surface of the star), in accordance with what can be deduced from the fact that $ln(\sqrt{A})$ behaves in this case like a potential of gravitation [4]. This limit obviously does not correspond to the same position as outside.

The particle slows down, then moves away from the star, recrosses the singularity and joined the limit $r = r_0$ outside: it can thus oscillate on both sides of the singularity.

To visualize more clearly, the simplest is to carry out a change of variable with the relation (6) (*cf.* figure 3). A logarithmic scale is needed and we see that it is usually almost certain that the particle reaches the surface of the star.

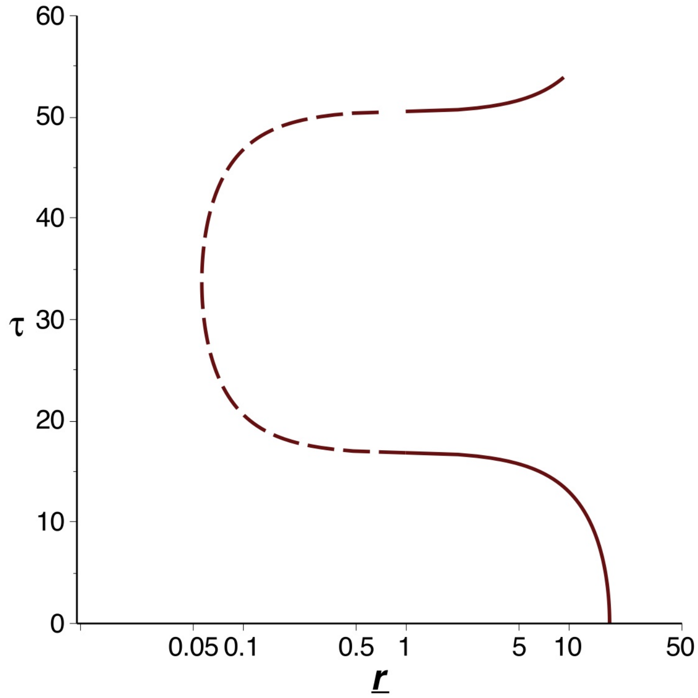

Fig. 3: variations (in reduced notations) of the proper duration of fall as a function of the "isotropic" radial variable.

One can integrate with the isotropic $\underline{r}$ coordinate; the equation (11) simplifies as for (13), but the equation is less simple and the accuracy of numerical integration is not so good:

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{c^2\ \underline{r}^4}{(\underline{r}+\underline{R}_s)^2}.\left(\frac{A_0}{(\underline{r}-\underline{R}_s)^2} - \frac{1}{(\underline{r}+\underline{R}_s)^2}\right)\ ; \tag{14}$$

$$\frac{d^2r}{d\tau^2} = -\frac{2c^2\underline{R}_s\ \underline{r}^3}{(\underline{r}+\underline{R}_s)^3}.\left(\frac{1}{(\underline{r}+\underline{R}_s)^2} + \frac{\underline{R}_s\ A_0}{(\underline{r}-\underline{R}_s)^3}\right). \tag{15}$$

One notes indeed a change of sign of the acceleration (with divergence) for the crossing of the singularity; it seems moreover that there is impossibility of equilibrium for $\underline{r}_0 = \underline{R}_s$ (A_0 tends to zero as $(\underline{r}_0 - \underline{R}_s)^2$ only and the second derivative diverges).

From these considerations rises obviously that, if one adopts the reinterpretation of metric suggested here, the reasoning developed in parts 3.4 and 3.5 of [1] are no longer suitable.

From another point of view, as stated in [1], the duration of the fall seems infinite for any outside observer (the gap between the particle and the singularity seems to decrease about exponentially). This brings up a paradox. One can indeed consider a particle in external circular orbit; the temporal dilation coefficient due to its movement is finished and does not change anything qualitatively. If another particle in vertical freefall crosses the one in rotation, it will return on the same level after a finished proper duration, but never for the second. This seems in contradiction with the principle of reciprocity.

However, it is likely that this circumstance is impossible for other reasons (for a complete demonstration, it would be necessary to study the interior field within the star and to specify the connection of the external field with the interior field, imposing the relation between the size of the star and that of the singularity). As indicated previously, it would be necessary to suppose $\underline{r}_0 \approx \underline{R}_s$ otherwise the particle in fall would reach the surface of the star and would not go out again. Now, there does not exist circular trajectories passing so close to the star (even the light finishes its trajectory in spiral if it passes too much close to the star; *cf.* figure 8 in [1]).

On the other hand, it appears that if there would exist a static black hole the matter on the surface of the star would inevitably undergo a field close to the one outside just above... that is to say a repulsive field. The matter on the surface would thus be ejected: there cannot exist any static black hole.

If a black hole is formed by collapse of a large amount of mass in a very reduced space (for example collision of two neutron stars), then the formed object can not be stable. The peripheral matter "in excess", responsible for the occurrence of the singularity, is ejected.

But, unlike what occurs for a single particle, this changes the metric: the singularity moves away and then disappears progressively as the peripheral matter is ejected; the formed unstable object most probably explodes in supernova and only the transitory phenomena are masked by a singularity.

Conclusion

The surprising discoveries of quantum mechanics may have accustomed us not to seek hardly the physical interpretation of the studied concepts. That perhaps diverted us a little from the rigor necessary to study precisely the coordinates being used to describe the objects of the type “black hole” (or other “wormholes”). Certainly, in general relativity, the coordinates are in principle arbitrary, but they still have to respect some rules: the geometric “maps” they allow us to define must be bijective. The effects on the physical properties which result from this can be fundamental: the gravitation can be repulsive and the static black holes cannot exist.

References

[1] C. Geuting, “Relativité de l’espace-temps, effets gravitationnels”, Bull. Un. Prof. Phys. Chim., vol. 107, n° 952, p. 279-299 (march 2013).

[2] As indicated in [1], one can consult: P. Tourrenc “Relativité et gravitation” (Paris, ed. Armand Colin, 1992); but if one does not have this book and does not wish to buy it solely for that, this particular method is exposed in few of other works (the use of the affine connection is the most frequent), it is thus useful to specify.

[3] L. Landau and E. Lifchitz, “The classical theory of fields”, (ed. Pergamon Press, 1971).

[4] S. Weinberg, “Gravitation and cosmology” (ed. Wiley, 1972).

[5] Strangely, this property seems totally disregarded; no general relativity work seems to indicate it (in 1975, during my DEA in theoretical physics, I had made this remark to my teacher M. A. Tonnelat, without being able to cause for that any interest).

[6] This unexpected property, consequence of the preceding one, appeared to me only in the years 1990, when I started to try to write an article on this subject (my attempts were vain: I did not convince any editor); my interest was raised again when I noticed that finally another physicist made the comment of this: F. Desombre, “La gravitation selon Einstein, (re)découvrir quelques trésors de la Relativité Générale”, Bull. Un. Prof. Phys. Chim., vol. 106, n° 949, p. 1185-1197 (december 2012).